

RIESZ BASES CONSISTING OF ROOT FUNCTIONS OF 1D DIRAC OPERATORS

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ABSTRACT. For one-dimensional Dirac operators

$$Ly = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + vy, \quad v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

subject to periodic or antiperiodic boundary conditions, we give necessary and sufficient conditions which guarantee that the system of root functions contains Riesz bases in $L^2([0, \pi], \mathbb{C}^2)$.

In particular, if the potential matrix v is skew-symmetric (i.e., $\overline{Q} = -P$), or more generally if $\overline{Q} = tP$ for some real $t \neq 0$, then there exists a Riesz basis that consists of root functions of the operator L .

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1. INTRODUCTION

We consider one-dimensional Dirac operators of the form

$$(1.1) \quad L_{bc}(v)y = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + v(x)y, \quad v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

with periodic matrix potentials v such that $P, Q \in L^2([0, \pi], \mathbb{C}^2)$, subject to periodic (Per^+) or antiperiodic (Per^-) boundary conditions (bc):

$$(1.2) \quad Per^+ : y(\pi) = y(0); \quad Per^- : y(\pi) = -y(0).$$

Our goal is to give necessary and sufficient conditions on potentials v which guarantee that the system of periodic (or antiperiodic) root functions of $L_{Per^\pm}(v)$ contains Riesz bases.

The free operators $L_{Per^\pm}^0 = L_{Per^\pm}(0)$ have discrete spectrum:

$$Sp(L_{Per^\pm}^0) = \Gamma^\pm, \quad \text{where} \quad \Gamma^\pm = \begin{cases} 2\mathbb{Z} & \text{if } bc = Per^+ \\ 2\mathbb{Z} + 1 & \text{if } bc = Per^- \end{cases}$$

and each eigenvalue is of multiplicity 2. The spectra of perturbed operators $L_{Per^\pm}(v) = L_{Per^\pm}^0 + v$ is also discrete; for $n \in \Gamma^\pm$ with large enough $|n|$ the perturbed operator has "twin" eigenvalues λ_n^\pm close to n . In the case where $\lambda_n^- \neq \lambda_n^+$ for large enough $|n|$, could the corresponding normalized "twin eigenfunctions" form a Riesz basis?

Recently, in the case of Hill operators, many authors focused on this problem (see [1, 2, 5, 6, 8, 10, 11, 12, 15, 16] and the bibliography there). It may happen

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that $\lambda_n^- \neq \lambda_n^+$ for $|n| > N_*$ but the system of normalized eigenfunctions fails to give a convergent eigenfunction expansion (see [2, Theorem 71]).

In the present paper we consider such a problem in the case of 1D periodic Dirac operators. In [7], we have singled out a class of potentials v which smoothness could be determined only by the rate of decay of related spectral gaps $\gamma_n = \lambda_n^+ - \lambda_n^-$, where λ_n^\pm are the eigenvalues of $L = L(v)$ considered on $[0, \pi]$ with periodic (for even n) or antiperiodic (for odd n) boundary conditions. This class X is determined by the properties of the functionals $\beta_n^-(v; z)$ and $\beta_n^+(v; z)$ (see below (2.8)) to be equivalent in the following sense: there are $c, N > 0$ such that

$$c^{-1}|\beta_n^+(v; z_n^*)| \leq |\beta_n^-(v; z_n^*)| \leq c|\beta_n^+(v; z_n^*)|, \quad |n| > N, \quad z_n^* = (\lambda_n^+ + \lambda_n^-)/2 - n.$$

Section 3 contains the main results of this paper. We prove that if $v \in X$ then the system of root functions of the operator $L_{Per^\pm}(v)$ contains Riesz bases in $L^2([0, \pi], \mathbb{C}^2)$. Theorem 3.1, which is analogous to Theorem 1 in [6] (or Theorem 2 in [5]), gives necessary and sufficient conditions for existence of such Riesz bases. Theorem 3.2 is a modification of Theorem 3.1 that is more suitable for application to concrete classes of potentials.

Applications of Theorems 3.1 and 3.2 are given in Section 4. In particular, we prove that if the potential matrix v is skew-symmetric (i.e., $\overline{Q} = -P$) then the system of root functions of $L_{Per^\pm}(v)$ contains Riesz bases in $L^2([0, \pi], \mathbb{C}^2)$.

2. PRELIMINARIES

1. Let H be a separable Hilbert space, and let $(e_\alpha, \alpha \in \mathcal{I})$ be an orthonormal basis in H . If $A : H \rightarrow H$ is an automorphism, then the system

$$(2.1) \quad f_\alpha = Ae_\alpha, \quad \alpha \in \mathcal{I},$$

is an unconditional basis in H . Indeed, for each $x \in H$ we have

$$x = A(A^{-1}x) = A\left(\sum_{\alpha} \langle A^{-1}x, e_\alpha \rangle e_\alpha\right) = \sum_{\alpha} \langle x, (A^{-1})^* e_\alpha \rangle f_\alpha = \sum_{\alpha} \langle x, \tilde{f}_\alpha \rangle f_\alpha,$$

i.e., (f_α) is a basis, its biorthogonal system is $\{\tilde{f}_\alpha = (A^{-1})^* e_\alpha, \alpha \in \mathcal{I}\}$, and the series converge unconditionally. Moreover, it follows that

$$(2.2) \quad 0 < c \leq \|f_\alpha\| \leq C, \quad m^2 \|x\|^2 \leq \sum_{\alpha} |\langle x, \tilde{f}_\alpha \rangle|^2 \|f_\alpha\|^2 \leq M^2 \|x\|^2,$$

with $c = 1/\|A^{-1}\|$, $C = \|A\|$, $M = \|A\| \cdot \|A^{-1}\|$ and $m = 1/M$.

A basis of the form (2.1) is called *Riesz basis*. One can easily see that the property (2.2) characterizes Riesz bases, i.e., a basis (f_α) is a Riesz bases if and only if (2.2) holds with some constants $C \geq c > 0$ and $M \geq m > 0$. Another characterization of Riesz bases is given by the following assertion (see [9, Chapter 6, Section 5.3, Theorem 5.2]): *If (f_α) is a normalized basis (i.e., $\|f_\alpha\| = 1 \forall \alpha$), then it is a Riesz basis if and only if it is unconditional.*

A countable family of bounded projections $\{P_\alpha : H \rightarrow H, \alpha \in \mathcal{I}\}$ is called *unconditional basis of projections* if $P_\alpha P_\beta = 0$ for $\alpha \neq \beta$ and

$$x = \sum_{\alpha \in \mathcal{I}} P_\alpha(x) \quad \forall x \in H,$$

where the series converge unconditionally in H .

If $\{H_\alpha, \alpha \in \mathcal{I}\}$ is a maximal family of mutually orthogonal subspaces of H and Q_α is the orthogonal projection on H_α , $\alpha \in \mathcal{I}$, then $\{Q_\alpha, \alpha \in \mathcal{I}\}$ is an unconditional basis of projections. A family of projections $\{P_\alpha, \alpha \in \mathcal{I}\}$ is called a *Riesz basis of projections* if there is a family of orthogonal projections $\{Q_\alpha, \alpha \in \mathcal{I}\}$ and an isomorphism $A : H \rightarrow H$ such that

$$(2.3) \quad P_\alpha = A Q_\alpha A^{-1}, \quad \alpha \in \mathcal{I}.$$

In view of (2.3), if $\{P_\alpha\}$ is a Riesz basis of projections, then there are constants $a, b > 0$ such that

$$(2.4) \quad a\|x\|^2 \leq \sum_{\alpha} \|P_\alpha x\|^2 \leq b\|x\|^2 \quad \forall x \in H.$$

For a family of projections $\mathcal{P} = \{P_\alpha, \alpha \in \mathcal{I}\}$ the following properties are equivalent (see [9, Chapter 6]):

- (i) \mathcal{P} is an unconditional basis of projections;
- (ii) \mathcal{P} is a Riesz basis of projections.

Lemma 2.1. *Let $(P_\alpha, \alpha \in \mathcal{I})$ be a Riesz basis of two-dimensional projections in a Hilbert space H , and let $f_\alpha, g_\alpha \in \text{Ran } P_\alpha$, $\alpha \in \mathcal{I}$ be linearly independent unit vectors. Then the system $\{f_\alpha, g_\alpha, \alpha \in \mathcal{I}\}$ is a Riesz basis if and only if*

$$(2.5) \quad \kappa := \sup |\langle f_\alpha, g_\alpha \rangle| < 1.$$

Proof. Suppose that the system $\{f_\alpha, g_\alpha, \alpha \in \mathcal{I}\}$ is a Riesz basis in H . Then

$$x = \sum_{\alpha} (f_\alpha^*(x)f_\alpha + g_\alpha^*(x)g_\alpha), \quad x \in H,$$

where f_α^*, g_α^* are the conjugate functionals. By (2.2), the one-dimensional projections

$$P_\alpha^1(x) = f_\alpha^*(x)f_\alpha, \quad P_\alpha^2(x) = g_\alpha^*(x)g_\alpha, \quad \alpha \in \mathcal{I},$$

are uniformly bounded. On the other hand, it is easy to see that

$$\|P_\alpha^1\|^2 \geq (1 - |\langle f_\alpha, g_\alpha \rangle|)^{-1}, \quad \|P_\alpha^2\|^2 \geq (1 - |\langle f_\alpha, g_\alpha \rangle|)^{-1},$$

so (2.5) holds.

Conversely, suppose (2.5) holds. Then we have for every $\alpha \in \mathcal{I}$

$$(1 - \kappa) (|f_\alpha^*(x)|^2 + |g_\alpha^*(x)|^2) \leq \|P_\alpha(x)\|^2 \leq (1 + \kappa) (|f_\alpha^*(x)|^2 + |g_\alpha^*(x)|^2)$$

which implies, in view of (2.4),

$$\frac{a}{1 + \kappa} \|x\|^2 \leq \sum_{\alpha} (|f_\alpha^*(x)|^2 + |g_\alpha^*(x)|^2) \leq \frac{b}{1 - \kappa} \|x\|^2.$$

Therefore, (2.2) holds, which means that the system $\{f_\alpha, g_\alpha, \alpha \in \mathcal{I}\}$ is a Riesz basis in H . \square

2. We consider the Dirac operator (1.1) with $bc = Per^\pm$ in the domain

$$\text{Dom } (L_{Per^\pm}(v)) = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : y_1, y_2 \text{ are absolutely continuous, } y(\pi) = \pm y(0) \right\}.$$

Then the operator $L_{Per^\pm}(v)$ is densely defined and closed; its adjoint operator is

$$(2.6) \quad (L_{Per^\pm}(v))^* = L_{Per^\pm}(v^*), \quad v^* = \begin{pmatrix} 0 & \overline{Q} \\ \overline{P} & 0 \end{pmatrix}.$$

Lemma 2.2. *The spectra of the operators $L_{Per^\pm}(v)$ are discrete. There is an $N = N(v)$ such that the union $\cup_{|n| > N} D_n$ of the discs $D_n = \{z : |z - n| < 1/4\}$ contains all but finitely many of the eigenvalues of L_{Per^+} and L_{Per^-} while the remaining finitely many eigenvalues are in the rectangle $R_N = \{z : |Re z|, |Im z| \leq N + 1/2\}$.*

Moreover, for $|n| > N$ the disc D_n contains two (counted with algebraic multiplicity) periodic (if n is even) or antiperiodic (if n is odd) eigenvalues λ_n^-, λ_n^+ such that $Re \lambda_n^- < Re \lambda_n^+$ or $Re \lambda_n^- = Re \lambda_n^+$ and $Im \lambda_n^- \leq Im \lambda_n^+$.

See details and more general results about localization of these spectra in [13, 14] and [2, Section 1.6].

Lemma 2.2 allows us to apply the Lyapunov–Schmidt projection method and reduce the eigenvalue equation $Ly = \lambda y$ for $\lambda \in D_n$ to an eigenvalue equation in the two-dimensional space $E_n^0 = \{L^0 Y = nY\}$ (see [2, Section 2.4]). This leads to the following (see in [2] the formulas (2.59)–(2.80) and Lemma 30).

Lemma 2.3. (a) *For large enough $|n|$, $n \in \mathbb{Z}$, there are functionals $\alpha_n(v, z)$ and $\beta_n^\pm(v, z)$, $|z| < 1$ such that a number $\lambda = n + z$, $|z| < 1/4$, is a periodic (for even n) or antiperiodic (for odd n) eigenvalue of L if and only if z is an eigenvalue of the matrix*

$$(2.7) \quad \begin{bmatrix} \alpha_n(v, z) & \beta_n^-(v, z) \\ \beta_n^+(v, z) & \alpha_n(v, z) \end{bmatrix}.$$

(b) *A number $\lambda = n + z^*$, $|z^*| < \frac{1}{4}$, is a periodic (for even n) or antiperiodic (for odd n) eigenvalue of L of geometric multiplicity 2 if and only if z^* is an eigenvalue of the matrix (2.7) of geometric multiplicity 2.*

The functionals $\alpha_n(z; v)$ and $\beta_n^\pm(z; v)$ are well defined for large enough $|n|$ by explicit expressions in terms of the Fourier coefficients $p(m)$, $q(m)$, $m \in 2\mathbb{Z}$ of the potential entries P, Q about the system $\{e^{imx}, m \in 2\mathbb{Z}\}$ (see [2, Formulas (2.59)–(2.80)]). Here we provide formulas only for $\beta_n^\pm(v; z)$:

$$(2.8) \quad \beta_n^\pm(v; z) = \sum_{\nu=0}^{\infty} \sigma_\nu^\pm \quad \text{with} \quad \sigma_0^+ = q(2n), \quad \sigma_0^- = p(-2n),$$

$$\sigma_\nu^+ = \sum_{j_1, \dots, j_{2\nu} \neq n} \frac{q(n + j_1)p(-j_1 - j_2)q(j_2 + j_3) \dots p(-j_{2\nu-1} - j_{2\nu})q(j_{2\nu} + n)}{(n - j_1 + z)(n - j_2 + z) \dots (n - j_{2\nu-1} + z)(n - j_{2\nu} + z)},$$

$$\sigma_\nu^- = \sum_{j_1, \dots, j_{2\nu} \neq n} \frac{p(-n - j_1)q(j_1 + j_2)p(-j_2 - j_3) \dots q(j_{2\nu-1} + j_{2\nu})p(-j_{2\nu} - n)}{(n - j_1 + z)(n - j_2 + z) \dots (n - j_{2\nu-1} + z)(n - j_{2\nu} + z)},$$

where $j_1, \dots, j_{2\nu} \in n + 2\mathbb{Z}$.

Next we summarize some basic properties of $\alpha_n(z; v)$ and $\beta_n^\pm(z; v)$.

Proposition 2.4. (a) *The functions $\alpha_n(z; v)$ and $\beta_n^\pm(z; v)$ depend analytically on z for $|z| \leq 1$. For $|n| \geq n_0$ the following estimates hold:*

$$(2.9) \quad |\alpha_n(v; z)|, |\beta_n^\pm(v; z)| \leq C \left(\mathcal{E}_{|n|}(r) + 1/\sqrt{|n|} \right), \quad |z| \leq 1/2;$$

$$(2.10) \quad \left| \frac{\partial \alpha_n}{\partial z}(v; z) \right|, \left| \frac{\partial \beta_n^\pm}{\partial z}(v; z) \right| \leq C \left(\mathcal{E}_{|n|}(r) + 1/\sqrt{|n|} \right), \quad |z| \leq 1/4,$$

where $r = (r(m))$, $r(m) = \max\{|p(\pm m)|, |q(\pm m)|\}$, $C = C(\|r\|)$, $n_0 = n_0(r)$ and

$$(\mathcal{E}_m(r))^2 = \sum_{|k| \geq m} |r(k)|^2.$$

(b) For large enough $|n|$, the number $\lambda = n + z$, $z \in D = \{\zeta : |\zeta| \leq 1/4\}$, is an eigenvalue of L_{Per^\pm} if and only if $z \in D$ satisfies the basic equation

$$(2.11) \quad (z - \alpha_n(z; v))^2 = \beta_n^+(z; v)\beta_n^-(z, v),$$

(c) For large enough $|n|$, the equation (2.11) has exactly two roots in D counted with multiplicity.

Proof. The assertion (a) is proved in [2, Proposition 35]. Lemma 2.3 implies (b). By (2.9), $\sup_D |\alpha_n(z)| \rightarrow 0$ and $\sup_D |\beta_n^\pm(z)| \rightarrow 0$ as $n \rightarrow \infty$. Therefore, (c) follows from the Rouché theorem. \square

In view of Lemma 2.2, for large enough $|n|$ the numbers $z_n^* = (\lambda_n^+ + \lambda_n^-)/2 - n$ are well defined. The following estimate of γ_n from above follows from (2.9) and (2.10) (see [2, Lemma 40]).

Lemma 2.5. *For large enough $|n|$,*

$$(2.12) \quad \gamma_n = |\lambda_n^+ - \lambda_n^-| \leq (1 + \delta_n)(|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|)$$

with $\delta_n \rightarrow 0$ as $|n| \rightarrow \infty$.

Remark. Here and sometimes thereafter, we suppress the dependence on v in the notations and write $\alpha_n(z)$ and $\beta_n^\pm(z)$.

3. In view of the above consideration, there is $n_0 = n_0(v)$ such that λ_n^\pm , $\beta_n^\pm(z)$ and $\alpha_n(z)$ are well-defined for $|n| > n_0$, and Lemmas 2.2, 2.3, 2.5 and Proposition 2.4 hold. Let us set

$$(2.13) \quad \mathcal{M}^\pm = \{n \in \Gamma^\pm : |n| > n_0, \lambda_n^- \neq \lambda_n^+\}.$$

Definition. Let X^\pm be the class of all Dirac potentials v with the following property: there are constants $c \geq 1$ and $N \geq n_0$ such that

$$(2.14) \quad \frac{1}{c}|\beta_n^+(v; z_n^*)| \leq |\beta_n^-(v; z_n^*)| \leq c|\beta_n^+(v; z_n^*)| \quad \text{if } n \in \mathcal{M}^\pm, |n| \geq N.$$

Lemma 2.6. *If $v \in X^\pm$ and the set \mathcal{M}^\pm is infinite, then for $n \in \mathcal{M}^\pm$ with sufficiently large $|n|$ we have*

$$(2.15) \quad \frac{1}{2}|\beta_n^\pm(v; z_n^*)| \leq |\beta_n^\pm(v; z)| \leq 2|\beta_n^\pm(v; z_n^*)| \quad \forall z \in K_n := \{z : |z - z_n^*| \leq \gamma_n\}.$$

Proof. By Lemma 2.5, if $v \in X^\pm$ then for $n \in \mathcal{M}^\pm$ with large enough $|n|$ we have $\beta_n^\pm(z_n^*) \neq 0$. In view of (2.10), if $z \in K_n$ then for large enough $|n|$

$$|\beta_n^\pm(z) - \beta_n^\pm(z_n^*)| \leq \varepsilon_n |z - z_n^*| \leq \varepsilon_n \gamma_n,$$

where $\varepsilon_n = C \left(\mathcal{E}_{|n|}(r) + 1/\sqrt{|n|} \right) \rightarrow 0$ as $|n| \rightarrow \infty$. By Lemma 2.5, for large enough $|n|$ we have $\gamma_n \leq 2(|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|)$. Then, for $n \in \mathcal{M}$,

$$|\beta_n^\pm(z) - \beta_n^\pm(z_n^*)| \leq 2\varepsilon_n (|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|) \leq 2\varepsilon_n(1 + c) |\beta_n^\pm(z_n^*)|$$

which implies, for sufficiently large $|n|$,

$$[1 - 2\varepsilon_n(1 + c)] |\beta_n^\pm(z_n^*)| \leq |\beta_n^\pm(z)| \leq [1 + 2\varepsilon_n(1 + c)] |\beta_n^\pm(z_n^*)|.$$

Since $\varepsilon_n \rightarrow 0$ as $|n| \rightarrow \infty$, (2.15) follows. \square

Proposition 2.7. *Suppose that $v \in X^\pm$ and the corresponding set \mathcal{M}^\pm is infinite. Then for $n \in \mathcal{M}^\pm$ with large enough $|n|$*

$$(2.16) \quad \frac{2\sqrt{c}}{1+4c} (|\beta_n^-(v; z_n^*)| + |\beta_n^+(v; z_n^*)|) \leq \gamma_n \leq 2 (|\beta_n^-(v; z_n^*)| + |\beta_n^+(v; z_n^*)|).$$

Proof. The estimate of γ_n from above follows from Lemma 2.5. By Lemma 2.5, for $n \in \mathcal{M}^\pm$ with large enough $|n|$ we have $\beta_n^\pm(z_n^*) \neq 0$. Set

$$t_n = |\beta_n^+(z_n^+)|/|\beta_n^-(z_n^+)|, \quad z_n^+ = \lambda_n^+ - n, \quad n \in \mathcal{M}^\pm.$$

By Lemma 2.6, t_n is well defined for large enough $|n|$. By Lemma 49 in [2], there exists a sequence $(\delta_n)_{n \in \mathbb{Z}}$ with $\delta_n \rightarrow 0$ as $|n| \rightarrow \infty$ such that, for $n \in \mathcal{M}^\pm$ with large enough $|n|$,

$$(2.17) \quad |\gamma_n| \geq \left(\frac{2\sqrt{t_n}}{1+t_n} - \delta_n \right) (|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|).$$

In view of (2.15) in Lemma 2.6, for large enough $|n|$ we have $1/(4c) \leq t_n \leq 4c$. Therefore, by (2.17) it follows

$$\gamma_n \geq \left(\frac{2\sqrt{4c}}{1+4c} - \delta_n \right) (|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|),$$

which implies (since $\delta_n \rightarrow 0$ as $|n| \rightarrow \infty$) the left inequality in (2.16). This completes the proof. \square

3. RIESZ BASES OF ROOT FUNCTIONS

In view of Lemma 2.2, the Dirac operators $L_{Per^\pm}(v)$ have discrete spectra; for N large enough and $n \in \Gamma^\pm$ with $|n| > N$ the Riesz projections

$$(3.1) \quad S_N^\pm = \frac{1}{2\pi i} \int_{\partial R_N} (z - L_{Per^\pm})^{-1} dz, \quad P_n^\pm = \frac{1}{2\pi i} \int_{|z-n|=\frac{1}{4}} (z - L_{Per^\pm})^{-1} dz$$

are well-defined and $\dim S_N^\pm < \infty$, $\dim P_n^\pm = 2$. Further we suppress in the notations the dependence on the boundary conditions Per^\pm and write S_N , P_n only. By [4, Theorem 3],

$$(3.2) \quad \sum_{n \in \Gamma^\pm, |n| > N} \|P_n - P_n^0\|^2 < \infty,$$

where P_n^0 are the Riesz projections of the free operator. Moreover, the Bari–Markus criterion implies (see Theorem 9 in [4]) that the spectral Riesz decompositions

$$(3.3) \quad f = S_N f + \sum_{n \in \Gamma^\pm, |n| > N} P_n f \quad \forall f \in L^2([0, \pi], \mathbb{C}^2)$$

converge unconditionally. In other words, $\{S_N, P_n, n \in \Gamma^\pm, |n| > N\}$ is a Riesz basis of projections in the space $L^2([0, \pi], \mathbb{C}^2)$.

Theorem 3.1. (A) *If $v \in X^\pm$, then there exists a Riesz basis in $L^2([0, \pi], \mathbb{C}^2)$ which consists of root functions of the operator $L_{Per^\pm}(v)$.*

(B) *If $v \notin X^\pm$, then the system of root functions of the operator $L_{Per^\pm}(v)$ does not contain Riesz bases.*

Remark. To avoid any confusion, let us emphasize that in Theorem 3.1 two *independent* theorems are stacked together: one for the case of periodic boundary conditions Per^+ , and another one for the case of antiperiodic boundary conditions Per^- .

Proof. We consider only the case of periodic boundary conditions $bc = Per^+$ since the proof is the same in the case of antiperiodic boundary conditions $bc = Per^-$.

(A) Fix $v \in X^+$, and let $N = N(v) > n_0(v)$ be chosen so large that Lemma 2.6, Proposition 2.7 and (3.1)–(3.3) holds for $|n| > N$.

If $n \notin \mathcal{M}^+$ then $\lambda_n^* = n + z_n^*$ is a double eigenvalue. In this case we choose $f(n), g(n) \in \text{Ran}(P_n)$ so that

$$(3.4) \quad \|f(n)\| = \|g(n)\| = 1, \quad L_{Per^+}(v)f(n) = \lambda_n^* f(n), \quad \langle f(n), g(n) \rangle = 0.$$

If $n \in \mathcal{M}^+$ then λ_n^- and λ_n^+ are simple eigenvalues. Now we choose corresponding eigenvectors $f(n), g(n) \in \text{Ran}(P_n)$ so that

$$(3.5) \quad \|f(n)\| = \|g(n)\| = 1, \quad L_{Per^+}(v)f(n) = \lambda_n^+ f(n), \quad L_{Per^+}(v)g(n) = \lambda_n^- g(n).$$

Let H be the closed linear span of the system

$$\Phi = \{f(n), g(n) : n \in \Gamma^+, |n| > N\}.$$

By (3.3), $L^2([0, \pi], \mathbb{C}^2) = H \oplus \text{Ran}(S_N)$. Since $\dim S_N < \infty$, the theorem will be proved if we show that the system Φ is a Riesz basis in the space H .

By (3.3), the system of two-dimensional projections $\{P_n : n \in \Gamma^+, |n| > N\}$ is Riesz basis of projections in H . By Lemma 2.1, the system Φ is a Riesz basis in H if and only if

$$\sup_{n \in \Gamma^+, |n| > N} |\langle f(n), g(n) \rangle| < 1.$$

By (3.4), we need to consider only indices $n \in \mathcal{M}^+$. Next we show that

$$(3.6) \quad \sup_{\mathcal{M}^+} |\langle f(n), g(n) \rangle| < 1.$$

By Lemma 2.6 the quotient $\eta_n(z) = \beta_n^-(z)/\beta_n^+(z)$ is a well defined analytic function on a neighborhood of the disc $K_n = \{z : |z - z_n^*| \leq \gamma_n\}$. Moreover, in view of (2.14) and (2.15), we have

$$(3.7) \quad \frac{1}{4c} \leq |\eta_n(z)| \leq 4c \quad \text{for } n \in \mathcal{M}^+, z \in K_n.$$

Since $\eta_n(z)$ does not vanish in K_n , there is an appropriate branch Log of $\log z$ (which depend on n) defined on a neighborhood of $\eta_n(K_n)$. We set

$$\text{Log}(\eta_n(z)) = \log |\eta_n(z)| + i\varphi_n(z);$$

then

$$(3.8) \quad \eta_n(z) = \beta_n^-(z)/\beta_n^+(z) = |\eta_n(z)|e^{i\varphi_n(z)},$$

so the square root $\sqrt{\beta_n^-(z)/\beta_n^+(z)}$ is a well defined analytic function on a neighborhood of K_n by

$$(3.9) \quad \sqrt{\beta_n^-(z)/\beta_n^+(z)} = \sqrt{|\eta_n(z)|}e^{\frac{i}{2}\varphi_n(z)}.$$

Now the basic equation (2.11) splits into the following two equations

$$(3.10) \quad z = \zeta_n^+(z) := \alpha_n(z) + \beta_n^+(z) \sqrt{\beta_n^-(z)/\beta_n^+(z)},$$

$$(3.11) \quad z = \zeta_n^-(z) := \alpha_n(z) - \beta_n^+(z) \sqrt{\beta_n^-(z)/\beta_n^+(z)}.$$

For large enough $|n|$, each of the equations (3.10) and (3.11) has exactly one root in the disc K_n . Indeed, in view of (2.10),

$$\sup_{|z| \leq 1/2} |d\zeta_n^\pm/dz| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, for large enough $|n|$ each of the functions ζ_n^\pm is a contraction on the disc K_n , which implies that each of the equations (3.10) and (3.11) has at most one root in the disc K_n . On the other hand, Lemma 2.2 implies that for large enough $|n|$ the basic equation (2.11) has exactly two simple roots in K_n , so each of the equations (3.10) and (3.11) has exactly one root in the disc K_n .

For large enough $|n|$, let $z_1(n)$ (respectively $z_2(n)$) be the only root of the equation (3.10) (respectively (3.11)) in the disc K_n . Of course, we have

$$\text{either (i) } z_1(n) = \lambda_n^- - n, \quad z_2(n) = \lambda_n^+ - n \quad \text{or (ii) } z_1(n) = \lambda_n^+ - n, \quad z_2(n) = \lambda_n^- - n.$$

Further we assume that (i) takes place; the case (ii) may be treated in the same way, and in both cases we have

$$(3.12) \quad |z_1(n) - z_2(n)| = \gamma_n = |\lambda_n^+ - \lambda_n^-|.$$

We set

$$(3.13) \quad f^0(n) = P_n^0 f(n), \quad g^0(n) = P_n^0 g(n).$$

From (3.2) it follows that $\|P_n - P_n^0\| \rightarrow 0$. Therefore,

$$\|f(n) - f^0(n)\| = \|(P_n - P_n^0)f(n)\| \leq \|P_n - P_n^0\| \rightarrow 0, \quad \|g(n) - g^0(n)\| \rightarrow 0,$$

so $|\langle f(n) - f^0(n), g(n) - g^0(n) \rangle| \rightarrow 0$. Since $\|f(n)\|^2 = \|f^0(n)\|^2 + \|f(n) - f^0(n)\|^2$ and $\langle f(n), g(n) \rangle = \langle f^0(n), g^0(n) \rangle + \langle f(n) - f^0(n), g(n) - g^0(n) \rangle$, we obtain

$$(3.14) \quad \|f^0(n)\|, \|g^0(n)\| \rightarrow 1, \quad \limsup_{n \rightarrow \infty} |\langle f(n), g(n) \rangle| = \limsup_{n \rightarrow \infty} |\langle f^0(n), g^0(n) \rangle|.$$

By Lemma 2.3, $f^0(n)$ is an eigenvector of the matrix $\begin{pmatrix} \alpha_n(z_1) & \beta_n^-(z_1) \\ \beta_n^+(z_1) & \alpha_n(z_1) \end{pmatrix}$ corresponding to its eigenvalue $z_1 = z_1(n)$, i.e.,

$$\begin{pmatrix} \alpha_n(z_1) - z_1 & \beta_n^-(z_1) \\ \beta_n^+(z_1) & \alpha_n(z_1) - z_1 \end{pmatrix} f^0(n) = 0.$$

Therefore, $f^0(n)$ is proportional to the vector $\left(\frac{z_1 - \alpha_n(z_1)}{\beta_n^+(z_1)}, 1 \right)^T$. Taking into account (3.8), (3.9) and (3.10) we obtain

$$(3.15) \quad f^0(n) = \frac{\|f^0(n)\|}{\sqrt{1 + |\eta_n(z_1)|}} \begin{pmatrix} \sqrt{|\eta_n(z_1)|} e^{\frac{i}{2}\varphi(z_1)} \\ 1 \end{pmatrix}.$$

In an analogous way, from (3.8), (3.9) and (3.11) it follows

$$(3.16) \quad g^0(n) = \frac{\|g^0(n)\|}{\sqrt{1 + |\eta_n(z_2)|}} \begin{pmatrix} -\sqrt{|\eta_n(z_2)|} e^{\frac{i}{2}\varphi(z_2)} \\ 1 \end{pmatrix}.$$

Now, (3.15) and (3.16) imply

$$(3.17) \quad \langle f^0(n), g^0(n) \rangle = \|f^0(n)\| \|g^0(n)\| \frac{1 - \sqrt{|\eta_n(z_1)|} \sqrt{|\eta_n(z_2)|} e^{i\psi_n}}{\sqrt{1 + |\eta_n(z_1)|} \sqrt{1 + |\eta_n(z_2)|}},$$

where

$$\psi_n = \frac{1}{2} [\varphi_n(z_1(n)) - \varphi_n(z_2(n))].$$

Next we explain that

$$(3.18) \quad \psi_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $\varphi_n = \text{Im} (\text{Log } \eta_n)$ we obtain, taking into account (3.12), that

$$|\varphi_n(z_1(n)) - \varphi_n(z_2(n))| \leq \sup_{[z_1, z_2]} \left| \frac{d}{dz} (\text{Log } \eta_n) \right| \cdot \gamma_n,$$

where $[z_1, z_2]$ denotes the segment with end points $z_1 = z_1(n)$ and $z_2 = z_2(n)$.

By (2.10) in Proposition 2.4 and (2.15) in Lemma 2.6 we estimate

$$\frac{d}{dz} (\text{Log } \eta_n) = \frac{1}{\beta_n^-(z)} \frac{d\beta_n^-(z)}{dz} - \frac{1}{\beta_n^+(z)} \frac{d\beta_n^+(z)}{dz}, \quad z \in [z_1, z_2],$$

as follows:

$$\left| \frac{d}{dz} (\text{Log } \eta_n) \right| \leq \frac{\varepsilon_n}{|\beta_n^-(z_n^*)|} + \frac{\varepsilon_n}{|\beta_n^+(z_n^*)|}$$

where $\varepsilon_n = C \left(\mathcal{E}_{|n|}(r) + \frac{1}{\sqrt{|n|}} \right) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, (2.14) and (2.16) imply that $|\varphi_n(z_1(n)) - \varphi_n(z_2(n))| \leq 4(1+c) \cdot \varepsilon_n \rightarrow 0$, i.e., (3.18) holds.

From (3.17) it follows

$$(3.19) \quad |\langle f^0(n), g^0(n) \rangle|^2 = \|f^0(n)\|^2 \|g^0(n)\|^2 \cdot \Pi_n,$$

with

$$(3.20) \quad \Pi_n = \frac{1 + |\eta_n(z_1)| |\eta_n(z_2)| - 2\sqrt{|\eta_n(z_1)| |\eta_n(z_2)|} \cos \psi_n}{(1 + |\eta_n(z_1)|) (1 + |\eta_n(z_2)|)}.$$

Now (3.18) implies $\cos \psi_n > 0$ for large enough n , so taking into account that $\|f^0(n)\|, \|g^0(n)\| \leq 1$, we obtain by (3.7)

$$|\langle f^0(n), g^0(n) \rangle|^2 \leq \Pi_n \leq \frac{1 + |\eta_n(z_1)| |\eta_n(z_2)|}{(1 + |\eta_n(z_1)|) (1 + |\eta_n(z_2)|)} \leq \delta < 1$$

with

$$\delta = \sup \left\{ \frac{1 + xy}{(1+x)(1+y)} : \frac{1}{4c} \leq x, y \leq 4c \right\}.$$

Finally, (3.14) shows that (3.6) holds, which completes the proof of (A).

(B) For every Dirac potential v we set

$$(3.21) \quad t_n(z) = \begin{cases} |\beta_n^-(z)/\beta_n^+(z)| & \text{if } \beta_n^+(z) \neq 0, \\ \infty & \text{if } \beta_n^+(z) = 0, \beta_n^-(z) \neq 0, \\ 1 & \text{if } \beta_n^+(z) = 0, \beta_n^-(z) = 0; \end{cases}$$

then $t_n(z)$, $|z| < 1$, is well-defined for large enough $|n|$.

If $v \notin X^+$, then there is a subsequence of indices (n_k) in \mathcal{M}^+ such that one of the following holds:

$$(3.22) \quad t_{n_k}(z_{n_k}^*) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

$$(3.23) \quad t_{n_k}(z_{n_k}^*) \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Next we consider only the case (3.22) because the case (3.23) could be handled in a similar way – if $1/t_{n_k}(z_{n_k}^*) \rightarrow 0$, then one may exchange the roles of β_n^+ and β_n^- and use the same argument.

In the above notations, if (3.22) holds then there is a sequence (τ_k) of positive numbers such that

$$(3.24) \quad t_{n_k}(z) \leq \tau_k \rightarrow 0 \quad \forall z \in [z_{n_k}^-, z_{n_k}^+],$$

where $[z_n^-, z_n^+]$ denotes the segment with end points z_n^- and z_n^+ .

Indeed, Lemma 2.5 and (3.22) imply that for large enough k

$$(3.25) \quad |\gamma_{n_k}| \leq 2(|\beta_{n_k}^-(z_{n_k}^*)| + |\beta_{n_k}^+(z_{n_k}^*)|) \leq 4|\beta_{n_k}^+(z_{n_k}^*)|.$$

In view of (2.10) in Proposition 2.4, for $z \in [z_n^-, z_n^+]$ and $n \in \mathcal{M}$ with large enough $|n|$ we have

$$(3.26) \quad |\beta_n^\pm(z) - \beta_n^\pm(z_n^*)| \leq \sup_{[z_n^-, z_n^+]} \left| \frac{\partial \beta_n^\pm}{\partial z}(z) \right| \cdot |z - z_n^*| \leq \varepsilon_n |\gamma_n|,$$

with $\varepsilon_n \rightarrow 0$ as $|n| \rightarrow \infty$. Therefore, from (3.25) and (3.26) it follows that

$$(3.27) \quad |\beta_{n_k}^+(z)| \geq |\beta_{n_k}^+(z_{n_k}^*)| - 4\varepsilon_{n_k} |\beta_{n_k}^+(z_{n_k}^*)| = (1 - 4\varepsilon_{n_k}) |\beta_{n_k}^+(z_{n_k}^*)|.$$

On the other hand, (3.25) and (3.26) imply that

$$|\beta_{n_k}^-(z)| \leq |\beta_{n_k}^-(z) - \beta_{n_k}^-(z_{n_k}^*)| + |\beta_{n_k}^-(z_{n_k}^*)| \leq 4\varepsilon_{n_k} |\beta_{n_k}^+(z_{n_k}^*)| + |\beta_{n_k}^-(z_{n_k}^*)|.$$

Thus, since $\varepsilon_{n_k} \rightarrow 0$, we obtain

$$\frac{|\beta_{n_k}^-(z)|}{|\beta_{n_k}^+(z)|} \leq \frac{4\varepsilon_{n_k} |\beta_{n_k}^+(z_{n_k}^*)| + |\beta_{n_k}^-(z_{n_k}^*)|}{(1 - 4\varepsilon_{n_k}) |\beta_{n_k}^+(z_{n_k}^*)|} = \frac{4\varepsilon_{n_k} + t_{n_k}(z_{n_k}^*)}{1 - 4\varepsilon_{n_k}} \rightarrow 0,$$

i. e., (3.24) holds with $\tau_k = \frac{4\varepsilon_{n_k} + t_{n_k}(z_{n_k}^*)}{1 - 4\varepsilon_{n_k}}$.

Let the vectors $f(n_k), g(n_k) \in \text{Ran}(P_{n_k})$ be chosen as in (3.5). Then $f(n_k)$ and $g(n_k)$ are unit eigenvectors which corresponds to the simple eigenvalues $\lambda_{n_k}^+$ and $\lambda_{n_k}^-$, so they are uniquely determined up to constant multipliers of absolute value one. Therefore, if the system of root functions of $L_{P_{er^+}}(v)$ contains Riesz bases, then the system $\{f(n_k), g(n_k) : k \in \mathbb{N}\}$ has to be a Riesz basis in its closed linear span which coincides with the closed linear span of $\{\text{Ran } P_{n_k}, k \in \mathbb{N}\}$. By Lemma 2.1 and (3.14), this would imply

$$(3.28) \quad \sup_k \langle f(n_k), g(n_k) \rangle = \sup_k \langle f^0(n_k), g^0(n_k) \rangle < 1.$$

Thus, the proof of (B) will be completed if we show that (3.28) fails.

By Lemma 2.3, $f^0(n_k)$ is an eigenvector of the matrix $\begin{pmatrix} \alpha_{n_k}(z_{n_k}^+) & \beta_{n_k}^-(z_{n_k}^+) \\ \beta_{n_k}^+(z_{n_k}^+) & \alpha_{n_k}(z_{n_k}^+) \end{pmatrix}$ corresponding to its eigenvalue $\lambda_{n_k}^+$, so it follows that $f^0(n)$ is proportional to the

vector $\begin{pmatrix} a(k) \\ 1 \end{pmatrix}$ with $a(k) = \frac{z_{n_k}^+ - \alpha_{n_k}(z_{n_k}^+)}{\beta_{n_k}^+(z_{n_k}^+)}$. Moreover, from (2.11), (3.21) and (3.24) it follows that

$$|a(k)| = \sqrt{t_{n_k}(z_{n_k}^+)} \leq \sqrt{\tau_k} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Therefore, we obtain

$$(3.29) \quad f^0(n_k) = \frac{\|f^0(n_k)\|}{\sqrt{|a(k)|^2 + 1}} \begin{pmatrix} a(k) \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{as } k \rightarrow \infty.$$

In the same we obtain that $g^0(n_k) \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as $k \rightarrow \infty$. Hence, $\langle f^0(n_k), g^0(n_k) \rangle \rightarrow 1$ as $k \rightarrow \infty$, so (3.28) fails, which completes the proof. \square

By Theorem 3.1, the condition (2.14) guarantees that there exists a Riesz basis in $L^2([0, \pi], \mathbb{C}^2)$ which consists of root functions of the operator $L_{Per^\pm}(v)$. Besides the case $v \in X_t$ (see the next section for a definition of the class of potentials X_t) it seems difficult to verify the condition (2.14). Moreover, since the points z_n^* are not known in advance, in order to check (2.14) one has to compare the values of $\beta_n^\pm(z)$ for all z close to 0. Next we give a modification of Theorem 3.1, which is more suitable for applications.

Consider potentials v such that for $n \in \Gamma^+ = 2\mathbb{Z}$ (or $n \in \Gamma^- = 2\mathbb{Z} + 1$) with large enough $|n|$

$$(3.30) \quad \beta_n^-(0) \neq 0, \quad \beta_n^+(0) \neq 0$$

and

$$(3.31) \quad \exists d > 0 : \quad d^{-1}|\beta_n^\pm(0)| \leq |\beta_n^\pm(z)| \leq d|\beta_n^\pm(0)| \quad \forall z \in D = \{z : |z| < 1/4\}.$$

Theorem 3.2. *Suppose $bc = Per^+$ (or $bc = Per^-$), and v is a Dirac potential such that (3.30) and (3.31) hold for $n \in \Gamma^+$ (respectively $n \in \Gamma^-$). Then*

(a) *the system of root functions of $L_{Per^+}(v)$ (respectively $L_{Per^-}(v)$) is complete and contains at most finitely many linearly independent associated functions;*

(b) *the system of root functions of $L_{Per^+}(v)$ (respectively $L_{Per^-}(v)$) contains Riesz bases if and only if*

$$(3.32) \quad 0 < \liminf_{n \in \Gamma^+} \frac{|\beta_n^-(0)|}{|\beta_n^+(0)|}, \quad \limsup_{n \in \Gamma^+} \frac{|\beta_n^-(0)|}{|\beta_n^+(0)|} < \infty$$

(or, respectively, \liminf and \limsup are taken over Γ^-).

Remark. Although the conditions (3.30)–(3.32) look too technical there is – after [2, 3] – a well elaborated technique to evaluate these parameters and check these conditions. To compare with the case of Hill operators with trigonometric polynomial coefficients – see [5, 6].

Proof. By Proposition 2.4, for large enough $|n|$ the basic equation

$$(3.33) \quad (z - \alpha_n(z))^2 = \beta_n^+(z)\beta_n^-(z),$$

has exactly two roots (counted with multiplicity) in the disc $D = \{z : |z| < 1/4\}$. Therefore, a number $\lambda = n + z$ with $z \in D$ is a periodic or antiperiodic eigenvalue of

algebraic multiplicity two if and only if $z \in D$ satisfies the system of two equations (3.33) and

$$(3.34) \quad 2(z - \alpha_n(z)) \frac{d}{dz} (z - \alpha_n(z)) = \frac{d}{dz} (\beta_n^+(z) \beta_n^-(z)).$$

In view of [4, Theorem 9], the system of root functions of the operator $L_{Per^\pm}(v)$ is complete, so Part (a) of the theorem will be proved if we show that there are at most finitely many n such that the system (3.33), (3.34) has a solution $z \in D$.

Suppose that $z^* \in D$ satisfies (3.33) and (3.34). By (2.10), for each $z \in D$

$$(3.35) \quad \left| \frac{d\alpha_n}{dz}(z) \right| \leq \varepsilon_n, \quad \left| \frac{d\beta_n^\pm}{dz}(z) \right| \leq \varepsilon_n \quad \text{with } \varepsilon_n \rightarrow 0 \text{ as } |n| \rightarrow \infty.$$

In view of (3.35), the equation (3.34) implies

$$2|z^* - \alpha_n(z^*)|(1 - \varepsilon_n) \leq \varepsilon_n (|\beta_n^+(z^*)| + |\beta_n^-(z^*)|).$$

By (3.33),

$$|z^* - \alpha_n(z^*)| = |\beta_n^+(z^*) \beta_n^-(z^*)|^{1/2},$$

so it follows, in view of (3.31),

$$2(1 - \varepsilon_n) \leq \varepsilon_n \left(\left| \frac{\beta_n^+(z^*)}{\beta_n^-(z^*)} \right|^{1/2} + \left| \frac{\beta_n^-(z^*)}{\beta_n^+(z^*)} \right|^{1/2} \right) \leq 2d\varepsilon_n.$$

Since $\varepsilon_n \rightarrow 0$ as $|n| \rightarrow \infty$, the latter inequality holds for at most finitely many n , which completes the proof of (a).

In view of (a), all but finitely many of the eigenvalues of L_{Per^\pm} are simple, i.e., $\lambda_n^- \neq \lambda_n^+$ for large enough $|n|$. One can easily see that Conditions (3.30)–(3.32) imply (2.14), respectively for $n \in \Gamma^+$ or $n \in \Gamma^-$, i.e., $v \in X^+$ or $v \in X^-$. Hence (b) follows from Theorem 3.1.

Remark. For Hill-Schrödinger operators with L^2 -potentials, an analog of Theorem 3.2 has been proven in [6, Theorem 1] (see also [5, Theorem 2]).

Theorem 3.1 gives a criterion for existence of Riesz basis consisting of root functions in the case of Dirac operators $L_{Per^\pm}(v)$ with L^2 -potentials. Technically its proof is based on the same argument as in [6, Theorem 1]. Moreover, analogs of Theorem 3.1 and 3.2 hold for Hill-Schrödinger operators with H^{-1} -potentials hold and the proofs are essentially the same. \square

4. APPLICATIONS

Consider the classes of Dirac potentials

$$(4.1) \quad X_t = \left\{ v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}, \quad Q(x) = t\overline{P(x)}, \quad P, Q \in L^2([0, \pi]) \right\}, \quad t \in \mathbb{R} \setminus \{0\}.$$

If $t = 1$ we get the class X_1 of symmetric Dirac potentials (which generate self-adjoint Dirac operators); X_{-1} is the class of skew-symmetric Dirac potentials. Next we show that if $v \in X_t$ then the system of root functions of $L_{Per^+}(v)$ or $L_{Per^-}(v)$ contains Riesz bases.

Proposition 4.1. *Suppose $v \in X_t$, $t \in \mathbb{R} \setminus \{0\}$.*

(a) If $t > 0$, then there is a symmetric potential \tilde{v} such that $L_{Per^\pm}(v)$ is similar to the self-adjoint operator $L_{Per^\pm}(\tilde{v})$, so its spectrum $Sp(L_{Per^\pm}(v)) \subset \mathbb{R}$.

(b) If $t < 0$, then there is a skew-symmetric potential \tilde{v} such that $L_{Per^\pm}(v)$ is similar to $L_{Per^\pm}(\tilde{v})$. Moreover, there is an $N = N(v)$ such that for $|n| > N$ either

(i) λ_n^- and λ_n^+ are simple eigenvalues and $\overline{\lambda_n^+} = \lambda_n^-$, $Im \lambda_n^\pm \neq 0$

or

(ii) $\lambda_n^+ = \lambda_n^-$ is a real eigenvalue of algebraic and geometric multiplicity 2.

(c) For large enough $|n|$

$$(4.2) \quad \overline{\beta_n^+(z_n^*, v)} = t \cdot \beta_n^-(z_n^*, v),$$

which implies $X_t \subset X^+ \cup X^-$.

(d) The system of root functions of $L_{Per^+}(v)$ (or $L_{Per^-}(v)$) contains Riesz bases.

Proof. For every $c \neq 0$, the Dirac operator $L_{Per^\pm}(v)$ is similar to the Dirac operator $L_{Per^\pm}(v_c)$ with $v_c = \begin{pmatrix} 0 & cP \\ \frac{1}{c}Q & 0 \end{pmatrix}$. Indeed, if $C = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$, then a simple calculation shows that $CL_{Per^\pm}(v) = L_{Per^\pm}(v_c)C$.

If $v \in X_t$ we set $\tilde{v} = v_c$ with $c = \sqrt{|t|}$. Then $\frac{1}{c}Q = \frac{t}{c}\overline{P} = \frac{t}{\sqrt{|t|}}\overline{P} = \pm c\overline{P}$. Therefore, \tilde{v} is symmetric or skew-symmetric, respectively, for $t > 0$ and $t < 0$.

(b) By (2.6), $(L_{Per^\pm}(v))^* = L_{Per^\pm}(v^*)$ with

$$v^* = \begin{pmatrix} 0 & \overline{Q} \\ \overline{P} & 0 \end{pmatrix} = \begin{pmatrix} 0 & tP \\ \frac{1}{t}Q & 0 \end{pmatrix} = v_t,$$

so the operator $L_{Per^\pm}(v)$ is similar to its adjoint operator. Therefore, if $\lambda \in Sp(L_{Per^\pm}(v))$, then $\overline{\lambda} \in Sp(L_{Per^\pm}(v))$ as well.

On the other hand by Lemma 2.2, there is an $N = N(v)$ such that for $|n| > N$ the disc $D_n = \{z : |z - n| < 1/4\}$ contains exactly two (counted with algebraic multiplicity) periodic (for even n) or antiperiodic (for odd n) eigenvalues of the operator L_{Per^\pm} . Therefore, if $\lambda \in D_n$ with $Im \lambda \neq 0$ is an eigenvalue of L_{Per^\pm} then $\overline{\lambda} \in D_n$ is also an eigenvalue of L_{Per^\pm} and $\overline{\lambda} \neq \lambda$, so λ and $\overline{\lambda}$ are simple, i.e., (i) holds.

Suppose $\lambda \in D_n$ is a real eigenvalue. If $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is a corresponding eigenvector, then passing to conjugates we obtain $L \begin{pmatrix} \overline{w_2} \\ -\overline{w_1} \end{pmatrix} = \lambda L \begin{pmatrix} \overline{w_2} \\ -\overline{w_1} \end{pmatrix}$, i.e., $\begin{pmatrix} \overline{w_2} \\ -\overline{w_1} \end{pmatrix}$ is also an eigenvector corresponding to the eigenvalue λ . But $\left\langle \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} \overline{w_2} \\ -\overline{w_1} \end{pmatrix} \right\rangle = 0$, so these vector-functions are linearly independent. Hence (ii) holds.

(c) By (i) and (ii) it follows that

$$z_n^* = \frac{1}{2}(\lambda_n^- + \lambda_n^+) - n \quad \text{is real for } |n| > N.$$

In view of (2.8), this implies that (4.2) holds.

(d) In view of (4.2), we have $v \in X$, so the claim follows from Theorem 3.1. \square

Example 4.2. *If a, b, A, B are non-zero complex numbers and*

$$(4.3) \quad v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \quad \text{with} \quad P(x) = ae^{2ix} + be^{-2ix}, \quad Q(x) = Ae^{2ix} + Be^{-2ix},$$

then the system of root functions of $L_{Per^+}(v)$ (or $L_{Per^-}(v)$) contains at most finitely many linearly independent associated functions. Moreover, the system of root functions of $L_{Per^+}(v)$ contains Riesz bases always, while the system of root functions of $L_{Per^-}(v)$ contains Riesz bases if and only if $|aA| = |bB|$.

Let us mention that if $bc = Per^+$ then it is easy to see by (2.8) that $\beta_n^\pm(z) = 0$ whenever defined, so the claim follows from Theorem 3.1.

If $bc = Per^-$, then the result follows from Theorem 3.2 and the asymptotics

$$(4.4) \quad \beta_n^+(0) = A^{\frac{n+1}{2}} a^{\frac{n-1}{2}} 4^{-n+1} \left[\left(\frac{n-1}{2} \right)! \right]^{-2} \left(1 + O(1/\sqrt{|n|}) \right),$$

$$(4.5) \quad \beta_n^-(0) = b^{\frac{n+1}{2}} B^{\frac{n-1}{2}} 4^{-n+1} \left[\left(\frac{n-1}{2} \right)! \right]^{-2} \left(1 + O(1/\sqrt{|n|}) \right).$$

Proofs of (4.4), (4.5) and similar asymptotics, related to other trigonometric polynomial potentials and implying Riesz basis existence or non-existence, will be given elsewhere (see similar results for the Hill-Schrödinger operator in [5, 6]).

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